The Calculations Behind a Gage Linearity Study

If you use your measurement system over a broad range of values, you may have to be concerned about the bias and linearity of the measurement system. The calculations behind a gage linearity study are fairly straightforward.   Software packages, like SPC for Excel, handle these calculations easily. This publication shows how the calculations are done to help you further understand the output from a gage linearity study.

In this issue:

Definitions of Bias and Linearity

We will use the **Measurement Systems Analysis** manual from the Automotive Industry Action Group (AIAG) to define bias and linearity. From the fourth edition of the manual:

Bias: The difference between the observed average of measurements (trials under repeatability conditions) and a reference value; historically referred to as *accuracy*. Bias is evaluated and expressed at a single point within the operating range of the measurement system.

Linearity: The difference in bias errors over the expected operating range of a measurement system. In other terms, linearity expresses the correlation of multiple and independent bias errors over the operating range.

Bias is how well your measurements compare to a reference value while linearity is assessing how accurate the measurements are over the operating range of the measurement system.

There have been two earlier SPC Knowledge Base articles on bias and linearity from 2007.

- [Variable Measurement Systems – Part 2: Bias](#)
- [Variable Measurement Systems - Part 3: Linearity](#)

This SPC Knowledge Base article updates some of the prior information with an increased focus on the calculations.

Example Data Set

A new measurement system will be used to monitor production for some quality characteristic. Five parts are selected from the process that represent the range of the quality characteristic in production.

The reference values of each part are determined. Then one operator measures each part ten times. The data are shown in Table 1.

Table 1: Example Data

Part	1	2	3	4	5
Ref Value	7	9	11	13	15
1	7.7	10.1	10.8	12.6	14.1
2	7.5	8.9	10.7	12.7	14.3
3	7.4	9.2	10.9	12.8	14.5
4	7.5	10.0	10.9	12.7	14.3
5	7.7	8.8	11.0	12.8	14.4
6	7.3	8.9	11.1	12.8	14.5
7	7.5	8.9	11.0	12.8	14.5
8	7.5	8.9	11.1	12.7	14.5
9	7.4	8.9	11.4	12.8	14.6
10	7.4	9.0	11.3	12.5	14.2

We will set g = the number of parts and m = the number of replications. We will use "i" to denote the part number and "j" to denote the replication. We will walk through the calculations beginning with bias.

Calculate the Bias

The first step is to calculate the bias for each individual reading. The bias for an individual reading is given by:

$$B_{ij} = x_{ij} - RV_i$$

where B_{ij} is the bias for the jth replication for part i, x_{ij} is the measurement value for the jth replication for part i, and RV_i is the reference value for part i.

The reference value for part 1 is given by RV_1, the reference value for part 2 is given by RV_2, and so on.

The bias for the first point in Table 1 is then the difference between the measurement value for the first replication (7.7) and the reference value for part 1 (7).

$$B_{11} = x_{11} - RV_1 = 7.7 - 7 = 0.7$$

Table 2 shows all the bias calculations.

Calculate the Average Bias per Part

The next step is to calculate the average bias for each part. The average bias for part i is given by:

$$\bar{B}_i = \frac{\sum_{j=1}^{m_i} B_{ij}}{m_i}$$

Table 2: Bias Calculations

Part	1	2	3	4	5
1	0.7	1.1	-0.2	-0.4	-0.9
2	0.5	-0.1	-0.3	-0.3	-0.7
3	0.4	0.2	-0.1	-0.2	-0.5
4	0.5	1	-0.1	-0.3	-0.7
5	0.7	-0.2	0.0	-0.2	-0.6
6	0.3	-0.1	0.1	-0.2	-0.5
7	0.5	-0.1	0.0	-0.2	-0.5
8	0.5	-0.1	0.1	-0.3	-0.5
9	0.4	-0.1	0.4	-0.2	-0.4
10	0.4	0.0	0.3	-0.5	-0.8
Sum	4.9	1.6	0.2	-2.8	-6.1
Average	0.49	0.16	0.02	-0.28	-0.61

where the summation is over the number of replications for part i. The averages for all 5 parts are also shown in Table 2.

With these calculations, you can now start the linearity chart. At this point, you can plot the individual bias results and the average bias for each part against the reference values. The resulting chart is shown in Figure 1.

Figure 1: Start of Linearity Chart
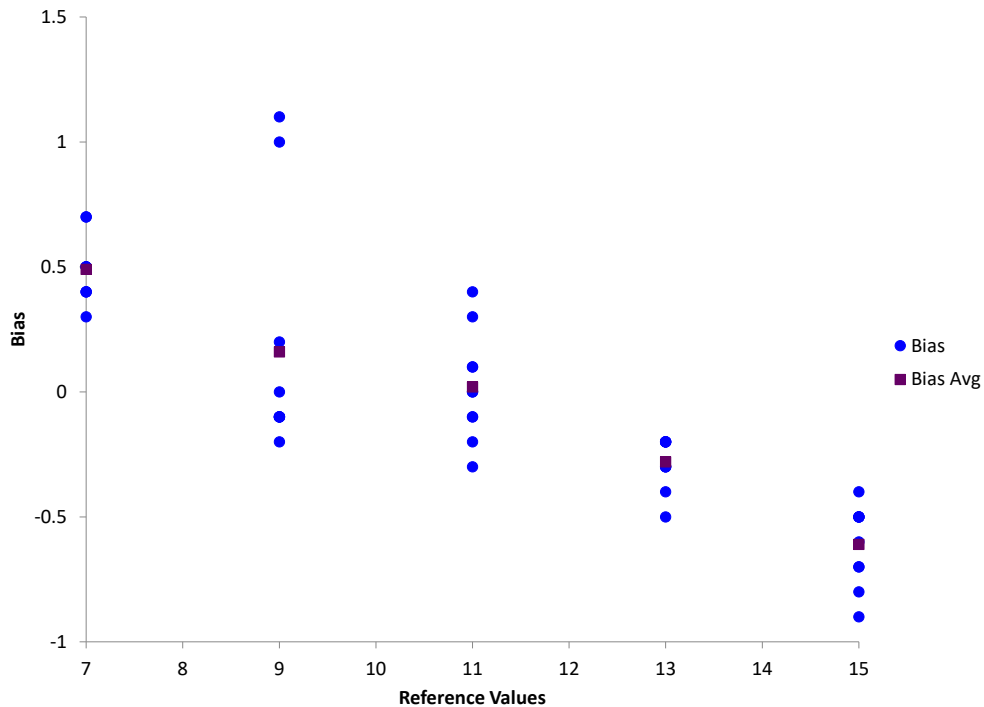
This chart shows that there is evidence of linearity in the measurement system. The bias is changing as the reference value increases.

Calculate the Best Fit Line Between Reference Values and Biases

The next step is to add the best-fit line between the reference values and the individual bias. So, you want to generate a regression line for the following data in Table 3:

Table 3: Regression Data

RV	Bias	RV	Bias	RV	Bias
7	0.7	9	-0.1	13	-0.2
7	0.5	9	-0.1	13	-0.2
7	0.4	9	0.0	13	-0.2
7	0.5	11	-0.2	13	-0.3
7	0.7	11	-0.3	13	-0.2
7	0.3	11	-0.1	13	-0.5
7	0.5	11	-0.1	15	-0.9
7	0.5	11	0.0	15	-0.7
7	0.4	11	0.1	15	-0.5
7	0.4	11	0.0	15	-0.7
9	1.1	11	0.1	15	-0.6
9	-0.1	11	0.4	15	-0.5
9	0.2	11	0.3	15	-0.5
9	1.0	13	-0.4	15	-0.5
9	-0.2	13	-0.3	15	-0.4
9	-0.1	13	-0.2	15	-0.8
9	-0.1	13	-0.3		

The reference values represent the x values and the average biases represent the y values in regression lingo. The equation being fitted then is:

$$y = b_o + b_1x$$

$$B = b_o + b_1(RV)$$

where b_0 = y intercept and b_1 = slope. You can use Excel's built in regression to determine the values of b_0 and b_1. You can also use the equations below to calculate the values.

$$b_1 = \frac{\sum(RV * B) - \frac{1}{gm}\sum RV \sum B}{\sum RV^2 - \frac{1}{gm}(\sum RV)^2}$$

$$b_0 = Average\ Bias - \ b_1(Average\ RV)$$

The following values can be used in the equations above:

- $\Sigma(RV*B)$ =- 77
- ΣRV = 550
- ΣB = -2.2
- $\Sigma(RV^2)$ = 6450
- g = 5
- m = 10
- Average Bias = -0.044
- Average RV = 11

The values of b_o and b_1 are then:

$$b_o = 1.408$$

$$b_1 = -0.132$$

Determine if the Slope is Significant

If there is no linearity in the bias results, i.e., the bias is the same at all reference values, you would expect b_1 to be zero. However, because of variation, you will rarely get $b_1 = 0$. The question you have to answer is:

Is the slope (b_1) significantly different from 0?

You can do this in a number of ways. One is to calculate the t-statistic and determine the p-value associated with that t-statistic.

The t-statistic is given by:

$$t = \frac{|b_1|}{\frac{s}{\sqrt{\Sigma(RV - Average\ RV)^2}}}$$

where s is the standard deviation, which is given by:

$$s = \sqrt{\frac{\Sigma B^2 - b_0 \Sigma B - b_1 \Sigma(RV * B)}{gm - 2}}$$

The only new term in the equation for s above is $\Sigma(B^2)$ which is 10.14. Putting the values into the above equation gives:

$$s = \sqrt{\frac{10.14 - 1.408(-2.2) + .132(-77)}{5(10) - 2}} = 0.253$$

You can calculate the t-statistic. The sum of the (RV – average RV)^2 for all measurements is 400 so the t-statistic is given by:

$$t = \frac{0.132}{(\frac{0.253}{\sqrt{400}})} = 10.43$$

The question you ask now is what is the probability you would get this t-statistic if the slope (linearity) is really 0? To find that, we can make use of the TDIST function in Excel. If you enter the following function TINV(|10.43|, 48, 2), it will give you the probability of getting a t value of 10.43 with 48 degrees of freedom when the slope is really zero. If this probability is less than 0.05, you will conclude the probability is small, so the slope must not be zero. In this example, the probability is very small (6.2E-14) so we conclude that the slope is not zero and the bias changes over the operating range of the measurement system.

Calculate the Confidence Interval Around the Best-Fit Line

The calculations now turn to the confidence limits around the regression line. The equations for calculating the lower and upper confidence bands are given below (for 95% confidence):

$$Lower\ Confidence\ Bound = \ b_0 + b_1 RV_0 - t_{(0.05, df)} s \left[\frac{1}{gm} + \frac{(RV_0 - Average\ RV)^2}{\sum(RV - Average\ RV)^2} \right]^{1/2}$$

$$Upper\ Confidence\ Bound = \ b_0 + b_1 RV_0 + \ t_{(0.05, df)} s \left[\frac{1}{gm} + \frac{(RV_0 - Average\ RV)^2}{\sum(RV - Average\ RV)^2} \right]^{1/2}$$

where RV_0 is a given reference value, e.g., 7. You will need to select several reference values over the range to create the line. All the variables in the above equations are defined given above. Using $RV_0 = 7$, gives the following results:

Lower Confidence Bound = 0.3594

Upper Confidence Bound = 0.60863

Create the Linearity Chart

You can now add three lines to the linearity chart: the best-fit line and the two confidence bounds. This is shown in Figure 2.

A fourth line where bias = 0 has also been added. The linearity chart presents clear picture of what is happening over the operating range for the measurement system. It is clear that the bias is changing. It is decreasing as the reference value goes up.

Summary

This publication showed the major calculations involved in a gage linearity study. The purpose of the study is to determine if the bias changes over the operating range of the measurement system. You can download an example of the linearity gage output from SPC for Excel at this link. This output gives more information, for example, which bias values are significantly different than zero.

Figure 2: Linearity Chart
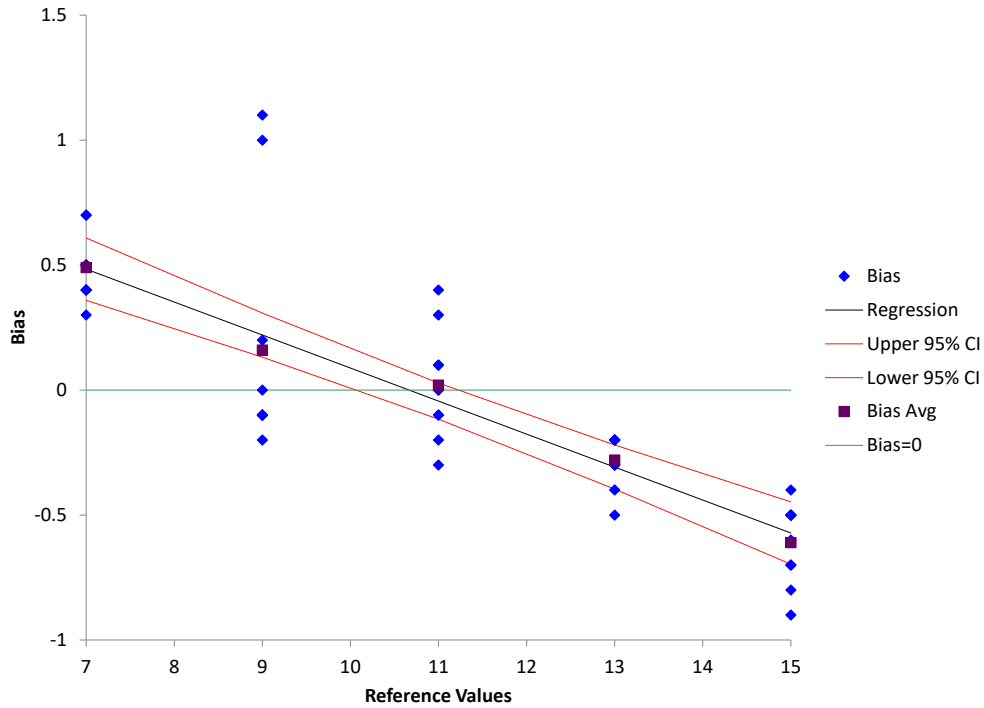
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Sincerely,

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